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A symmetry result for the p-Laplacian in a punctured manifold

Alberto Enciso [a], Daniel Peralta-Salas [b,*]

[a] *Depto. de Física Teórica II, Universidad Complutense, 28040 Madrid, Spain*
[b] *Dpto. de Matemáticas, Universidad Carlos III, 28911 Leganés, Madrid, Spain*

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ABSTRACT

Let M be a Riemannian manifold such that its geodesic spheres centered at a point $a \in M$ are isoperimetric and the distance function $\operatorname{dist}(\cdot, a)$ is isoparametric, and let $\Omega \subset M$ be a bounded domain. We prove that if there exists a lower bounded nonconstant function u which is p-harmonic $(1 < p \leqslant n)$ in the punctured domain $\Omega \setminus \{a\}$ such that both u and $\frac{\partial u}{\partial \nu}$ are constant on $\partial\Omega$, then u is radial and $\partial\Omega$ is a geodesic sphere. The proof hinges on a combination of maximum principles, isoparametricity and the isoperimetric inequality.

1. Introduction and statement of results

Since its appearance in Lord Rayleigh's classic treatise [28], the study of symmetry in overdetermined free boundary problems has become a major field of research in PDE theory, of great interest to both analysts and differential geometers. A major breakthrough in this field was the introduction of the celebrated moving planes method in two seminal articles by Alexandrov and Serrin [2,32]. Under the appropriate regularity conditions, this technique allows to establish that if there exists a nonconstant function u satisfying an elliptic equation

$$\operatorname{div}\big(a(u, |\nabla u|)\nabla u\big) = f\big(u, |\nabla u|\big)$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ and such that u and $\frac{\partial u}{\partial \nu}$ are constant on $\partial\Omega$, then $\partial\Omega$ is a round sphere and u is radially symmetric.

Further refinements of this idea have resulted in a wealth of related results allowing for less stringent regularity assumptions on the boundary and applying also to degenerate elliptic equations (cf. e.g. [7,23,27] and references therein). Whereas the moving planes method is the most widely used technique to obtain symmetry results, it is certainly not the only available one; a brief but enlightening discussion of different techniques can be found in Ref. [19].

The analysis of symmetry for free boundary problems on manifolds has received comparatively little attention due to the additional technical difficulties that appear in this context. Essentially the only technique available to tackle such problems is Serrin's method of moving planes, which has been successfully implemented in the hyperbolic space $\mathbb{H}^n$ and the hemisphere $\mathbb{S}^n_+$ [21,25]. The problem is still wide open for arbitrary Riemannian manifolds, and even the aforementioned extensions to constant curvature spaces are not at all straightforward. Quite remarkably, overdetermined problems in curved spaces arise naturally in the study of the event horizons of static black holes [5].

In this paper we aim to prove a symmetry result for the p-Laplacian

$$\Delta_p u := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big),$$

* Corresponding author.
E-mail addresses: aenciso@fis.ucm.es (A. Enciso), dperalta@math.uc3m.es (D. Peralta-Salas).

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on a certain class of Riemannian manifolds to be specified later. Here div and ∇ denote the divergence and gradient operators associated with the Riemannian structure. As in Refs. [13,16,34], we assume that $1 < p \leqslant n$ in order to ensure that there do not exist bounded p-harmonic functions with nonremovable singularities.

We consider weak solutions bounded from below to the overdetermined problem

$$\Delta_p u = 0 \quad \text{in } \Omega\backslash\{a\},$$
$$u|_{\partial\Omega} = c_1, \qquad |\nabla u||_{\partial\Omega} = c_2 > 0, \tag{1}$$

where c_1, c_2 are constants and Ω is a bounded, connected subset of an n-manifold M. We do not impose any a priori regularity conditions on $\partial\Omega$. By a weak solution we mean a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega\backslash\{a\})$ such that

$$\int_\Omega |\nabla u|^{p-2}\langle\nabla u, \nabla\phi\rangle\,\mathrm{d}x = 0$$

for every $\phi \in C_0^\infty(\Omega\backslash\{a\})$. Similarly, the boundary conditions are assumed to hold in the sense [12] that for all $\epsilon > 0$ there exists a neighborhood $U_\epsilon \supset \partial\Omega$ such that $|u(x) - c_1| < \epsilon$ and $||\nabla u(x)| - c_2| < \epsilon$ for a.e. $x \in U_\epsilon \cap \Omega$.

We shall always assume that M is a (possibly incomplete) smooth n-manifold such that, roughly speaking, its geodesic spheres centered at a are both isoperimetric and isoparametric. More precisely, we assume that:

(i) The geodesic ball $B(a,r)$ centered at a of radius r is an isoperimetric domain [26,30] for any $r > 0$, i.e., it is area-minimizing in the sense that the boundary of any domain with volume $|B(a,r)|$ has an area not smaller than $|\partial B(a,r)|$.
(ii) The distance function $\rho = \mathrm{dist}(a,\cdot)$ is isoparametric [38], i.e., it is of class C^2 in $M\backslash\{a\}$ and there exists a function $f \in C^0((0,\mathrm{diam}(M)))$ such that $\Delta\rho = f(\rho)$.

Remark 1. The prime example of such a manifold is the hemisphere $\mathbb{S}^n_+$. As discussed in Section 3, these spaces also include the Euclidean and hyperbolic spaces and some surfaces of nonconstant curvature. Let us also recall that, for any Riemannian manifold M, $\rho \in C^\infty(M\backslash(C(a)\cup\{a\}))$, where $a \in M$ and $C(a)$ denotes the cut locus of a. One should notice that condition (ii) implies that M is contractible.

Problem (1) arises in nonlinear potential theory: when $p = 2$, our main theorem stated below simply asserts that the electric field on a conducting hypersurface enclosing a point charge is constant if and only if the conductor is a sphere centered at the charge. When $p = 2$, the above problem on a Riemannian manifold appears in Physics as Electrostatics on anisotropic media [8], whereas for general $p > 1$ it arises in hydrodynamics, in the context of incompressible non-Newtonian fluids [11]. In Euclidean space, this problem has been recently studied [9] using a combination of the maximum principle and integral identities for P-functions which does not extend to curved manifolds.

The main result of the paper is the following

Theorem 2. *If conditions* (i) *and* (ii) *are satisfied, there exists a weak solution bounded from below to the overdetermined problem* (1) *if and only if Ω is a geodesic ball centered at a, implying that u is radial.*

We have not found in the literature any other symmetry result for manifolds of nonconstant curvature. We shall present the proof of this theorem, which does *not* make use of the moving planes method, in the following section. The final section of this paper completes the presentation of the main result with some examples of spaces satisfying conditions (i) and (ii) above and a discussion on the relations among isoperimetricity, isoparametricity and symmetry.

2. Proof of the main theorem

First of all, one should notice that the free boundary problem (1) is not readily amenable to a treatment via moving planes as, in principle, the isometry group of our manifold need not be sufficiently large to use this technique. In Euclidean space, Serrin's method can be successfully adapted to prove the desired result, but even in this case it is rather nontrivial due to the presence of a nonremovable singularity at a [1].

The idea of the proof of Theorem 2 is the following. We use the fact that the geodesic distance function to the point a is isoparametric to construct a model radial solution u^* which satisfies the overdetermined boundary problem in a geodesic ball Ω^* of arbitrary radius. If we impose that this radius be $\mathrm{dist}(a,\partial\Omega)$, we can compare the solution to the free boundary problem (1) and u^*, so that a combination of several maximum principles and the isoperimetric inequality shows that they must actually coincide, yielding that $\Omega = \Omega^*$.

By adding a constant to the solution u and dilating both u and the metric of the manifold, it can be easily seen that the actual values of the constants c_1 and $c_2 > 0$ are irrelevant. Hence we shall hereafter set $c_1 = 0$ and $c_2 = 1$, which allows us to restrict our attention to nonnegative solutions of Eq. (1).

Lemma 3. *Let $\Omega^* := B(a, R) \subset M$ be the geodesic ball centered at a and of radius R, and suppose $1 < p \leqslant n$. Then there exists a positive solution $u^* \in C^\infty(\Omega^* \backslash \{a\})$ to the overdetermined problem*

$$\Delta_p u^* = 0 \quad \text{in } \Omega^* \backslash \{a\},$$
$$u|_{\partial\Omega^*} = 0, \qquad \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega^*} = -1, \tag{2}$$

and this solution is radial.

Proof. Let $\rho = \operatorname{dist}(a, \cdot)$. On account of condition (ii) in the previous section and the fact that the gradient of the distance function has unit norm in $M \backslash \{a\}$, the p-Laplacian of a smooth function $\Psi(\rho)$ can be readily computed to be

$$\Delta_p \Psi(\rho) = \Psi'(\rho)^{p-2}\big[(p-1)\Psi''(\rho) + f(\rho)\Psi'(\rho)\big].$$

Using this equation it is not difficult to show that the radial function

$$u^*(x) = \int_{\rho(x)}^{R} \exp\left(\int_s^R \frac{f(t)}{p-1}\,\mathrm{d}t\right) \mathrm{d}s \tag{3}$$

yields the desired solution. Actually, its p-harmonicity follows from the latter identity, as is the fact that it is positive and satisfies the overdetermined boundary conditions is obvious from Eq. (3). □

In the proof of the theorem we shall invoke the following two lemmas. They are probably standard, but we sketch their proofs for completeness.

Lemma 4. *Under the hypotheses of Theorem 2, the domain Ω is of class C^2 and there exists a tubular neighborhood U of its boundary such that $u \in C^{1,\alpha}_{\text{loc}}(\overline{\Omega} \backslash \{a\}) \cap C^2(U \cap \Omega)$.*

Proof. When $p \neq 2$, the degeneracy of the p-Laplacian at the critical points of u implies that u is in general only $C^{1,\alpha}_{\text{loc}}(\Omega \backslash \{a\})$ [3,22,36], whereas it is $C^{2,\alpha}$ in a neighborhood of any of its regular points as a consequence of the standard elliptic regularity theory [14]. However, the weak boundary condition $|\nabla u| = 1$ ensures that there exists a tubular neighborhood U of $\partial\Omega$ such that Eq. (1) is elliptic in $U \cap \Omega$, so that $\partial\Omega$ is of class C^2 as a consequence of a theorem of Vogel [12,37].[1] In turn, a result of Lieberman [24] now implies that $u \in C^{1,\alpha}_{\text{loc}}(\overline{\Omega} \backslash \{a\})$. □

Lemma 5. *Let u be a solution of the overdetermined problem* (1)*, with $1 < p \leqslant n$. Then $-\Delta_p u = |\partial\Omega|\delta_a$ in the sense of distributions.*

Proof. As u has an isolated singularity at a, a theorem of Serrin [31] shows that the distributional Laplacian of u satisfies

$$-\Delta_p u = K\delta_a, \tag{4}$$

where δ_a stands for the delta function supported at a. Consider an open set U as in Lemma 4 and let $V \supset \partial\Omega$ be a proper open subset of U. Choose a $\phi_+ \in C_0^\infty(\Omega)$ with $\phi_+|_{\Omega\backslash U} = 1$ and $\operatorname{supp}\phi_+ \subset \Omega \backslash \overline{V}$, and define $\phi_- := 1 - \phi_+$. From Eqs. (1) and (4) and the fact that u is classical solution in U with $\nabla u = -\nu$ on $\partial\Omega$ by Lemma 4, it readily follows that

$$0 = \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla 1\rangle\,\mathrm{d}x = \int_\Omega |\nabla u|^{p-2}\big\langle \nabla u, \nabla(\phi_+ + \phi_-)\big\rangle\,\mathrm{d}x = K - |\partial\Omega|,$$

as claimed. □

We are now ready to tackle the proof of the main theorem.

Proof of Theorem 2. Set $R := \operatorname{dist}(a, \partial\Omega)$ and let $\Omega^* := B(a, R)$ be the largest ball centered at a and contained in Ω. Since a is an interior point of Ω and $\partial\Omega$ is a C^2 submanifold, the boundary $\partial\Omega^*$ must be tangent to $\partial\Omega$ at at least one point. We denote by u^* the radial function constructed in Lemma 3.

Since u tends to $+\infty$ at a by Ref. [31], the strong maximum principle for degenerate elliptic equations [6,15] ensures that u is strictly positive in Ω. Lemma 5 asserts that

$$-\Delta_p u = |\partial\Omega|\delta_a, \qquad -\Delta_p u^* = |\partial\Omega^*|\delta_a.$$

[1] $\partial\Omega$ is in fact $C^{2,\alpha}$ from each side [37], but we shall not need this refinement.

As $|\Omega^*| \leqslant |\Omega|$ and Ω^* saturates the isoperimetric inequality, it follows that $|\partial\Omega| \geqslant |\partial\Omega^*|$ and that the inequality only holds when $\Omega = \Omega^*$.

We shall now suppose $\Omega \neq \Omega^*$ and see that this assumption leads to a contradiction. Asymptotic results of Friedman, Kichenassamy and Véron (the proof presented in [10,20] for the Euclidean case carries over directly to the present situation) ensure that u behaves near the singularity as

$$u = \begin{cases} C_{n,p}|\partial\Omega|^{\frac{1}{p-1}}\rho^{-\frac{n-p}{p-1}} + o(\rho^{-\frac{n-p}{p-1}}) & \text{if } n \neq p, \\ C_{n,n}|\partial\Omega|^{\frac{1}{n-1}}\log\frac{1}{\rho} + o(\log\rho) & \text{if } n = p, \end{cases}$$

and similarly for u^*. Here $C_{n,p}$ are positive universal constants that only depend on n and p. In particular, from this it stems that $u > u^*$ in a neighborhood of a as by assumption $|\partial\Omega| > |\partial\Omega^*|$. Since moreover $u \geqslant u^*$ in $\partial\Omega^*$, it follows from the comparison principle [15] that $u > u^*$ in $\Omega^*\backslash\{a\}$ (note that $\nabla u^* \neq 0$ in $\Omega^*\backslash\{a\}$ and hence the strong version of the comparison principle holds, cf. [6]).

Let Γ be a connected component of the set $\partial\Omega \cap \partial\Omega^*$, which is known to be nonempty by the tangency condition. It is clear that Γ is a closed set. Let us consider the open tubular neighborhood U of $\partial\Omega$ defined in Lemma 4 and let W be the component of $U \cap \Omega^*$ whose closure contains Γ. On account of the boundary conditions we can take U small enough so that the gradient of u and u^* does not vanish in W.

As remarked in the proof of Lemma 4, Eqs. (1) and (2) are actually elliptic in W as a consequence of the boundary condition on the normal derivative of u. An argument similar to those in Refs. [32,37] shows that the function $v := u - u^*$ satisfies a linear elliptic equation of second order in this domain. Indeed, since u is of class C^2 in W the mean value theorem shows that there exist a continuous vector field X and a continuous tensor field T of type $(1,2)$ in W such that

$$|\nabla u|^{p-2} - |\nabla u^*|^{p-2} = \langle X, \nabla v\rangle,$$
$$|\nabla u|^{p-3}u_i u_j - |\nabla u^*|^{p-3}u_i^* u_j^* = T_{ij}^k v_k.$$

Here the subscripts denote covariant derivatives and the indices are raised and lowered using the metric tensor g. Subtracting the equations $\Delta_p u = \Delta_p u^* = 0$ and using these tensor fields one readily finds that

$$a^{ij}v_{ij} + b^i v_i = 0, \tag{5}$$

where

$$a^{ij} = |\nabla u|^{p-2}g^{ij} + (p-2)|\nabla u|^{p-3}u^i u^j,$$
$$b^i = X^i\Delta u^* + (p-2)T_{jk}^i(u^*)^{jk}.$$

The tensor a^{ij} being positive definite, Eq. (5) is the desired elliptic equation.

As v is nonnegative in $\overline{W}$, $v = \frac{\partial v}{\partial \nu} = 0$ in Γ and v satisfies the elliptic equation (5) in W, it follows from the Hopf boundary lemma [17] that $v = 0$ in W. W being open, this means that there exists a point $x \in \partial\Omega^*\backslash\partial\Omega$ such that $u(x) = 0$, contradicting the fact that u is strictly positive in Ω. Hence it follows that $\Omega = \Omega^*$ and $u = u^*$. □

3. Examples and discussion

In this paper we have proved a symmetry result for certain domains in spaces whose geodesic spheres are isoparametric and isoperimetric. Hence we shall start this section with some examples of spaces satisfying this hypothesis. It should be noted that, since both the isoperimetric problem and the characterization of isoparametric functions on manifolds are wide open today [30,35], an exhaustive list of these spaces cannot be given, and the number of known examples is likely to keep growing.

Example 1. The hemisphere $\mathbb{S}^n_+$ and the Euclidean and hyperbolic spaces $\mathbb{R}^n$ and $\mathbb{H}^n$ [4,35] (and sufficiently small balls centered at a in non-simply connected spaces of constant curvature).

Example 2. Planes with rotational symmetry with respect to a whose Gauss curvature is strictly decreasing from this point. The isoperimetric condition is proved in Ref. [29], while the fact that the distance function is isoparametric stems from the rotational symmetry.

While the proof of Theorem 2 is based on a combination of the properties of isoperimetric and isoparametric functions, it should be highlighted that the simultaneous verification of both conditions is not required for the existence of a solution to the overdetermined problem (1), as shown in the following

Example 3. Set $f(r) := r(1+r^2)$ and let us consider the plane with the metric given by $\mathrm{d}s^2 = \mathrm{d}r^2 + f(r)^2\,\mathrm{d}\theta^2$ in polar coordinates. A simple calculation shows that

$$u(r,\theta) = \frac{R(1+R^2)}{2}\log\frac{1+r^{-2}}{1+R^{-2}}$$

satisfies Eq. (1) in the punctured ball $\{0 < r < R\}$. On the other hand, the Gauss curvature of this space is given by

$$K(r) = -\frac{f''(r)}{f(r)} = -\frac{6}{1+r^2},$$

which is a strictly increasing function of the geodesic coordinate r. Under this assumption Ritoré has proved [29] that the geodesic spheres are not stable (in fact, there are no isoperimetric sets in this manifold), showing that the overdetermined problem (1) can be solved in domains that are not isoperimetric. Note that Theorem 2 does not apply here, so that uniqueness is not granted in this case.

It should be mentioned that, in dimension higher than 2, isoparametricity does not imply spherical symmetry. In particular, one can easily use Eq. (3) to produce examples of geodesic balls without spherical symmetry which admit solutions to the overdetermined problem (1) provided that the volume density function in normal coordinates depend only on the geodesic distance to a.

For manifolds M which do not satisfy either condition (i) or condition (ii) the existence of solutions to the overdetermined problem (1) on noncontractible domains Ω is wide open. If $p = 2$, it can be readily proved that if such solutions exist, they cannot be isoparametric, as we shall show next. Let us briefly recall that a function f on a Riemannian manifold M is isoparametric [38] if it is of class C^2 and there exist functions $\psi \in C^2(\mathbb{R})$ and $\phi \in C^0(\mathbb{R})$ which satisfy

$$|\nabla f|^2 = \psi(f), \qquad \Delta f = \phi(f).$$

Proposition 6. *Let M be an arbitrary n-manifold and let Ω be an open subset of M such that there exists an isoparametric solution u to the overdetermined problem* (1) *with $p = 2$. Then Ω is contractible.*

Proof. Since u is isoparametric in $\Omega\backslash\{a\}$, it follows [38] that the level sets of u near a are geodesic spheres. We shall show that the gradient of u does not vanish in $\Omega\backslash\{a\}$. In order to see this, note that the critical set of u must have codimension at least 2 as an immediate consequence of Holmgren's theorem [18]. Let S be a connected component of the critical set of u. Again by Wang's theorem [38], S is a smooth submanifold and the neighboring level sets of u must be tubes around S, contradicting the fact that u does not have any local extrema inside Ω by the maximum principle. Since the time-t flow of the vector field $X := -\frac{\nabla u}{|\nabla u|^2}$ maps $u^{-1}(c)$ onto $u^{-1}(c-t)$ diffeomorphically, it follows that all the level sets of u (including $\partial\Omega$) are geodesic spheres, proving the statement. $\square$

The above proof is specific to the case $p = 2$, since for general p the solution is generally not C^2 at its singular set, so that Wang's results for isoparametric functions are not applicable.

The connection between domains in which the overdetermined problem can be solved and hypersurfaces of constant mean curvature is necessarily subtle. It is well known that both the regular level sets of isoparametric functions and the boundaries of isoperimetric domains possess constant mean curvature. A critical inspection of the literature reveals that the notion of isoperimetricity usually arises in connection with the question of uniqueness (cf. the proof of Theorem 2), whereas isoparametric functions can be conveniently used to deal with existence (cf. Lemma 3 and Example 3, see also Ref. [33]). Along the lines of the preceding discussion, we find it would be of interest to prove or disprove the following

Conjecture 7. *Let us suppose that the distance function to a point a in an n-manifold M is isoparametric* (*in particular M is contractible*)*. Then the only compact domains which admit a solution to the overdetermined problem* (1) *are the geodesic balls.*

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